[2] $y = \sum_{n=0}^{\infty} a_n x^n$

$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$

$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$

$(1+x^3)y'' - 6xy$

$= \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=2}^{\infty} n(n-1)a_n x^{n+1} - \sum_{n=0}^{\infty} 6a_n x^{n+1}$

$= \sum_{n=-1}^{\infty} (n+3)(n+2)a_{n+3} x^{n+1} + \sum_{n=2}^{\infty} n(n-1)a_n x^{n+1} - \sum_{n=0}^{\infty} 6a_n x^{n+1}$

$= 2(1)a_2 + 3(2)a_3 x + 4(3)a_4 x^2$

$\quad - 6a_0 x - 6a_1 x^2$

$\quad + \sum_{n=2}^{\infty} [(n+3)(n+2)a_{n+3} + (n(n-1)-6)a_n] x^{n+1}$

$= 2a_2 + (6a_3 - 6a_0)x + (12a_4 - 6a_1)x^2$

$\quad + \sum_{n=2}^{\infty} [(n+3)(n+2)a_{n+3} + (n^2-n-6)a_n] x^{n+1} = 0$

$2a_2 = 0 \rightarrow a_2 = 0$

$6a_3 - 6a_0 = 0 \rightarrow a_3 = a_0$

$12a_4 - 6a_1 = 0 \rightarrow a_4 = \frac{1}{2}a_1$

$(n+3)(n+2)a_{n+3} + (n-3)(n+2)a_n = 0 \rightarrow a_{n+3} = -\frac{n-3}{n+3}a_n, n \geq 2$

LET $a_0 = 1, a_1 = 0 \rightarrow a_4 = 0 = a_7 = a_{10} = a_{13} = \ldots$

$a_2 = 0 \rightarrow a_5 = 0 = a_8 = a_{11} = a_{14} = \ldots$

$a_3 = a_0 = 1$

$n=3: a_6 = -\frac{0}{6}a_3 = 0 = a_9 = a_{12} = a_{15} = \ldots$

$y_1 = 1 + x^3$

LET $a_0 = 0,\ a_1 = 1$

$\hookrightarrow a_3 = 0 = a_6 = a_9 = a_{12} = \cdots$

$a_2 = 0 \rightarrow a_5 = 0 = a_8 = a_{11} = a_{14} = \cdots$

$a_4 = \frac{1}{2} a_1 = \frac{1}{2}$

$n=4:\ a_7 = -\frac{1}{7} a_4 = -\frac{1}{7 \cdot 2}$ $\quad k=2 \longrightarrow 3k+1 = 7$

$n=7:\ a_{10} = -\frac{4}{10} a_7 = \frac{4 \cdot 1}{(10 \cdot 7) 2}$ $\quad k=3 \longrightarrow 3k+1 = 10$

$n=10:\ a_{13} = -\frac{7}{13} a_{10} = -\frac{\cancel{7} \cdot 4 \cdot 1}{(13 \cdot 10 \cdot \cancel{7}) \cdot 2}$ $\quad k=4 \longrightarrow 3k+1 = 13$

$n=13:\ a_{16} = -\frac{10}{16} a_{13} = \frac{\cancel{10} \cdot 4 \cdot 1}{(16 \cdot 13 \cdot \cancel{10}) 2}$ $\quad k=5 \longrightarrow 3k+1 = 16$

$$y_2 = x + \frac{1}{2}x^4 + \sum_{k=2}^{\infty} (-1)^{k+1} \frac{4}{2(3k+1)(3k-2)} x^{3k+1}$$

$$= x + \frac{1}{2}x^4 + \sum_{k=2}^{\infty} (-1)^{k+1} \frac{2}{(3k+1)(3k-2)} x^{3k+1}$$

$k=0 \rightarrow$ TERM IS $(-1)^1 \frac{2}{1(-2)} x^1 = x$

$k=1 \rightarrow$ TERM IS $(-1)^2 \frac{2}{4(1)} x^4 = \frac{1}{2}x^4$

$$= \sum_{k=0}^{\infty} (-1)^{k+1} \frac{2}{(3k+1)(3k-2)} x^{3k+1}$$

$$y = c_1(1+x^3) + c_2 \sum_{k=0}^{\infty} (-1)^{k+1} \frac{2}{(3k+1)(3k-2)} x^{3k+1}$$

[3] $y'' + \frac{2x+3}{2x} y' - \frac{1}{2x^2} y = 0$

$\frac{2x+3}{2x}, -\frac{1}{2x^2}$ NOT CONT @ $x=0$

SO $x=0$ IS A SINGULAR POINT

$$\lim_{x\to 0} (x-0)\frac{2x+3}{2x} = \lim_{x\to 0} \frac{2x+3}{2} = \frac{3}{2}$$

$$\lim_{x\to 0} (x-0)^2\left(-\frac{1}{2x^2}\right) = \lim_{x\to 0} -\frac{1}{2} = -\frac{1}{2}$$

SO $x=0$ IS A REGULAR SINGULAR POINT

$$y = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

$$2x^2 y'' + (2x^2+3x)y' - y$$

$$= \sum_{n=0}^{\infty} 2(n+r)(n+r-1)a_n x^{n+r}$$

$$+ \sum_{n=0}^{\infty} 2(n+r) a_n x^{n+r+1} + \sum_{n=0}^{\infty} 3(n+r) a_n x^{n+r} - \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$= \sum_{n=0}^{\infty} 2(n+r)(n+r-1)a_n x^{n+r} + \sum_{n=0}^{\infty} 3(n+r)a_n x^{n+r} - \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$+ \sum_{n=1}^{\infty} 2(n+r-1) a_{n-1} x^{n+r}$$

$$= 2r(r-1)a_0 x^r + 3ra_0 x^r - a_0 x^r$$

$$+ \sum_{n=1}^{\infty} \left[(2(n+r)(n+r-1) + 3(n+r) - 1) a_n + 2(n+r-1)a_{n-1} \right] x^{n+r} = 0$$

$$2r(r-1)a_0 + 3ra_0 - a_0 = 0$$

$$(2r^2 - 2r + 3r - 1)a_0 = 0$$

$$2r^2 + r - 1 = 0 \longrightarrow (2r-1)(r+1) = 0 \longrightarrow r = \tfrac{1}{2}, -1$$

$r=-1: \ (2(n-1)(n-2)+3(n-1)-1)a_n + 2(n-2)a_{n-1} = 0, \quad n \geq 1$

$(2n^2-6n+4+3n-3-1)a_n + 2(n-2)a_{n-1} = 0$

$(2n^2-3n)a_n + 2(n-2)a_{n-1} = 0$

$$a_n = -\frac{2(n-2)}{n(2n-3)}a_{n-1}$$

LET $a_0 = 1$

$n=1: \ a_1 = -\frac{2(-1)}{1(-1)}a_0 = -2$

$n=2: \ a_2 = -\frac{2(0)}{2(1)}a_1 = 0 \longrightarrow a_3 = 0 = a_4 = a_5 = a_6 = \cdots$

$y_1 = x^{-1}(1-2x) = \frac{1}{x} - 2$

$r=\frac{1}{2}: \ (2(n+\frac{1}{2})(n-\frac{1}{2})+3(n+\frac{1}{2})-1)a_n + 2(n-\frac{1}{2})a_{n-1} = 0, \quad n \geq 1$

$(2n^2-\frac{1}{2}+3n+\frac{3}{2}-1)a_n + (2n-1)a_{n-1} = 0$

$(2n^2+3n)a_n + (2n-1)a_{n-1} = 0$

$$a_n = -\frac{2n-1}{n(2n+3)}a_{n-1}$$

LET $a_0 = 1$

$n=1: \ a_1 = -\frac{1}{1(5)}a_0 = -\frac{1}{5}$

$n=2: \ a_2 = -\frac{3}{2(7)}a_1 = \frac{3}{2(5 \cdot 7)}$

$n=3: \ a_3 = -\frac{5}{3(9)}a_2 = -\frac{3 \cdot \cancel{5}}{(2 \cdot 3)(\cancel{5} \cdot 7 \cdot 9)}$

$n=4: \ a_4 = -\frac{7}{4(11)}a_3 = \frac{3 \cdot \cancel{7}}{(2 \cdot 3 \cdot 4)(\cancel{7} \cdot 9 \cdot 11)}$

$$y_2 = x^{\frac{1}{2}}\left(1 - \frac{1}{5}x + \sum_{n=2}^{\infty}(-1)^n \frac{3}{n!(2n+1)(2n+3)}x^n\right)$$

$n=0 \rightarrow$ TERM IS $(-1)^0 \frac{3}{0!(1)(3)}x^0 = 1$

$n=1 \rightarrow$ TERM IS $(-1)^1 \frac{3}{1!(3)(5)}x^1 = \frac{1}{5}x^5$

$$y_2 = x^{\frac{1}{2}} \sum_{n=0}^{\infty}(-1)^n \frac{3}{n!(2n+1)(2n+3)}x^n$$